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THE STABILITY OF PERIODIC SOLUTIONS OF DISCONTINUOUS SYSTEMS THAT INTERSECT SEVERAL SURFACES OF DISCONTINUITY†

A. P. IVANOV

Moscow

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Systems of differential equations with discontinuous right-hand sides are considered, specifically investigating periodic solutions which simultaneously intersect two or more surfaces of discontinuity. It is shown that the Poincaré mapping along phase trajectories of the system in the neighbourhood of a fixed point, corresponding to periodic motion, is in general piecewise-differentiable: this neighbourhood divides into several sectors in which the Jacobians are different. For such mappings, theorems of stability in the first approximation [1] are not applicable, and one has to devise new stability criteria. Several necessary conditions for stability are obtained, as well as sufficient conditions. The results are used to investigate symmetric modes of motion of a vibro-impact system with two impact pairs. © 1999 Elsevier Science Ltd. All rights reserved.

The method of investigating stability in the first approximation was previously applied to discontinuous systems for solutions that intersect one surface of discontinuity [2]. It turned out that under such conditions the Poincaré mapping is differentiable, so that Lyapunov's theorems could be used.

1. FORMULATION OF THE PROBLEM

Linearization is one of the most effective approaches to investigating the stability of periodic motions of dynamical systems. The procedure consists of the following two steps:

1. Construction and solution of variational equations in the neighbourhood of the solution being investigated.

2. Verification of the stability conditions.

Let $x^*(t)$ be a τ-periodic solution of the system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\ t),\quad \mathbf{x},\ \mathbf{F} \in R^n \tag{1.1}$$

where the right-hand side is 2π-periodic in t (the period τ of the solution is a multiple of 2π).

If the function $\mathbf{F}$ is continuously differentiable in some neighbourhood of the trajectory $x^*(t)$, one can use the Poincaré–Lyapunov theory. The fundamental solution matrix $\mathbf{Y}(t_0, t)$ is defined by the equation

$$\dot{\mathbf{Y}}(t_0,\ t) = \mathbf{F}_\mathbf{x}(\mathbf{x}^*(t),\ t)\mathbf{Y}(t_0,\ t),\quad \mathbf{Y}(t_0,\ t_0) = \mathbf{E}_n \tag{1.2}$$

where $\mathbf{F}_\mathbf{x}$ is the Jacobian. The sufficient condition for stability is that all the eigenvalues of the monodromy matrix $\mathbf{Y}(t_0, t_0 + \tau)$ (that is, the Floquet multipliers) lie inside the unit circle in the complex plane; the necessary condition is that they lie either inside or on the circle [1].

Note that a given periodic motion may be associated with a fixed point of the Poincaré mapping φ: $R^n \to R^n$ along solutions of system (1.1) for the section $t = t_0 = (\mathrm{mod}\ \tau)$. This mapping is differentiable and its Jacobian is the monodromy matrix $\mathbf{Y}(t_0, t_0 + \tau)$.

If system (1.1) has a discontinuous right-hand side, additional technical difficulties arise. Attention has been devoted [2] to the case in which the vector-valued function $\mathbf{F}(\mathbf{x}, t)$ is discontinuous on a smooth surface $f(\mathbf{x}, t) = 0$, but both $\mathbf{F}$ and $\mathbf{F}_\mathbf{x}$ are continuous on either side of the surface, right up to the surface itself. It was assumed in that study that the solution intersects the surface of discontinuity without tangency. It was shown that, if the initial time t_0 does not correspond to a crossing of the surface of discontinuity, then the mapping φ is differentiable and its Jacobian can be constructed by "matching" the solutions of system (1.2) at the crossing time t' of the surface of discontinuity, according to the formula

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$$\mathbf{Y}(t_0,\ t'+0)=\mathbf{B}\mathbf{Y}(t_0,\ t'-0) \tag{1.3}$$

where

$$\mathbf{B}=\mathbf{E}_n+((\mathbf{F}^-,\ \mathbf{grad}\, f)+\partial f/\partial t)^{-1}(\mathbf{F}^+-\mathbf{F}^-)(\mathbf{grad}\, f)^T$$

The superscript T denotes transposition and $\mathbf{E}_n$ is the identity matrix.

After calculating the monodromy matrix, the stability problem is solved in exactly the same way as in the smooth case.

Another type of discontinuity is characteristic for systems with impact: when the phase trajectory reaches the boundary of the domain of continuous motion (which is defined by one or more inequalities of the form $f(\mathbf{x}) \geqslant 0$), it experiences a jump discontinuity in accordance with the formula

$$\mathbf{x}^+ = \mathbf{x}^- + \mathbf{I}(\mathbf{x}^-) \tag{1.4}$$

where the minus and plus superscripts correspond to the beginning and end of the impact and $\mathbf{I}$ is the impulse.

This type of periodic motion may be investigated by analogy with the previous case. The jump of the fundamental solution matrix upon impact is described by formula (1.3) with

$$\mathbf{B}=\mathbf{E}_{2n}+\mathbf{I}_x+(\mathbf{F}^-,\ \mathbf{grad}\, f)^{-1}(\mathbf{F}^+-\mathbf{F}^--\mathbf{I}_x\mathbf{F}^-)(\mathbf{grad}\, f)^T \tag{1.5}$$

In this case, too, the Poincaré mapping is still differentiable, despite the discontinuous nature of the trajectories.

In this paper we will discuss the stability of periodic trajectories that intersect several surfaces of discontinuity at the same time. Problems of this kind may be encountered when investigating systems of variable structure [3, 4], as well as mechanical systems with several impact pairs, among which there are no rigid constraints [5].

2. VARIATION OF THE SOLUTIONS INTERSECTING SEVERAL SURFACES OF DISCONTINUITY

Let us assume that the right-hand side of system (1.1) has discontinuities on the surfaces

$$f_j(\mathbf{x}) = 0\ (j = 1, ..., k) \tag{2.1}$$

All these surfaces are assumed to be smooth (that is, continuously differentiable, without singular points). They divide the phase space into several domains Ω_s, $s = 1, \ldots, r$, and it is assumed that at points where two or more surfaces intersect their normal vectors are linearly independent. The right-hand side of system (1.1) is continuously differentiable in each of the closed domains $\bar{\Omega}_s \times R$.

Consider a solution $\mathbf{x}^*(t)$ which at time $t = t'$ intersects two surfaces, $f_1(\mathbf{x}) = 0$ and $f_2(\mathbf{x}) = 0$. We will assume that the intersection occur without tangency, that is, the vectors $\dot{\mathbf{x}}^*(t' \pm 0)$ are transverse to both surfaces. Let $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}^*(t)$ denote the variation of the solution. We will investigate its behaviour near a point of discontinuity. The neighbourhood of the point is divided by the surfaces of discontinuity

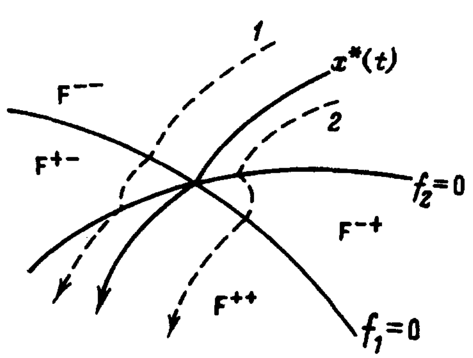

Fig. 1.

into four parts, in each of which the function $\mathbf{F}$ is defined differently (Fig. 1). Let us denote the corresponding continuous components by $\mathbf{F}^{--}$, $\mathbf{F}^{-+}$, $\mathbf{F}^{+-}$ and $\mathbf{F}^{++}$, where the minus and plus superscripts indicate the half-spaces in which the unperturbed trajectory is situated before and after intersecting the surfaces of discontinuity.

As is well known, the following estimates hold in regions where the right-hand side of system (1.1) is continuous

$$\mathbf{y}(t) = \mathbf{Y}(t_0, t)\mathbf{y}(t_0) + O(\|\mathbf{y}(t_0)\|^2) \tag{2.2}$$

where $\mathbf{Y}(t_0, t)$ is a solution of system (1.2). This relationship enables us to determine the times at which the perturbed trajectory intersects the surfaces of discontinuity. Let $\bar{\mathbf{x}}(t')$ be a perturbed trajectory for the equations of motion with right-hand side $\mathbf{F}^{--}$, and let $t' + \Delta_j t'$ be the times at which the trajectory intersects the surfaces $f_j = 0$ $(j = 1, 2)$.

Define quantities $\Delta_{1,2}t$ by the formulae

$$0 = f_j(\bar{\mathbf{x}}(t' + \Delta_j t)) = (\operatorname{grad} f_j, \mathbf{y}(t')) + (\operatorname{grad} f_j, \dot{\mathbf{x}}^*)\Delta_j t + O(\Delta_j t)^2$$

Consequently

$$\Delta_j t = (\operatorname{grad} f_j\big|_{\mathbf{x}^*(t')}, \mathbf{y}(t'))/V_j + O(\|\mathbf{y}\|^2), \quad V_j = -(\operatorname{grad} f_j, \mathbf{F}^{--}(\mathbf{x}^*(t'), t')) \tag{2.3}$$

Depending on the initial perturbation $\mathbf{y}(t_0)$, one may have inequalities $\Delta_1 t < \Delta_2 t$ or $\Delta_1 t > \Delta_2 t$. In the first case, the perturbed trajectory will first intersect the surface $f_1 = 0$ and then the surface $f_2 = 0$. If the inequality sign is reversed, the order of the intersections is also reversed (Fig. 1).

The jumps of the fundamental solution matrix are determined using formula (1.3). One must take into account the fact that the perturbed trajectory may intersect the surfaces of discontinuity either simultaneously or consecutively, in either order. The results may be summarized in the following proposition.

Proposition 1. If $\mathbf{x}^*(t)$ is a solution of system (1.1), which at time $t = t'$ intersects two surfaces of discontinuity (2.1) of the right-hand side without tangency, then estimate (2.2) remains valid at times $t > t'$, where the fundamental solution matrix experiences a discontinuity in accordance with the following formulae

$$\mathbf{B} = \begin{cases} \mathbf{B}_2(\mathbf{F}^{+-}, \mathbf{F}^{++})\mathbf{B}_1(\mathbf{F}^{--}, \mathbf{F}^{+-}), & \Delta_2 t < \Delta_1 t \\ \mathbf{B}_1(\mathbf{F}^{-+}, \mathbf{F}^{++})\mathbf{B}_2(\mathbf{F}^{--}, \mathbf{F}^{-+}), & \Delta_2 t > \Delta_1 t \\ \mathbf{B}_0 & \Delta_2 t = \Delta_1 t \end{cases} \tag{2.4}$$

where

$$\mathbf{B}_j(\mathbf{u}, \mathbf{v}) = \mathbf{E}_n + (\mathbf{u}, \operatorname{grad} f_j)^{-1}(\mathbf{v} - \mathbf{u})(\operatorname{grad} f_j)^T, \quad j = 1, 2$$

$$\mathbf{B}_0 = \mathbf{B}_1(\mathbf{F}^{--}, \mathbf{F}^{++})$$

This proposition may be extended to the case in which the trajectory simultaneously intersects $k \geqslant 3$ surfaces of discontinuity: there will then be a greater number of versions in formulae (2.4). The order in which the perturbed trajectory intersects the surfaces of discontinuity (2.1) will depend on the quantities $\Delta_j t$ in formulae (2.3), $j = 1, \ldots, k$. The first surface to be intersected is that for which this quantity is a minimum. In order to determine which surface is intersected next, the variation of the right-hand sides of the system when one of the surfaces of discontinuity is intersected must be taken into account. Substituting this changed expression into (2.3), we can calculate the quantities $\Delta_j^{(1)}t$, where j takes all values from 1 to k except for the subscript that corresponds to the first surface intersected. The minimum of these subscripts $\Delta_j^{(1)}t$ indicates the subscript that corresponds to the second surface intersected, and so on.

A generalization of another kind concerns systems with unilateral constraints. In such systems the trajectories do not intersect the surfaces (2.1), but when one of these surfaces is reached the trajectory experiences a discontinuity in accordance with formula (1.4).

Note that in the general case, when impact occurs against two or more constraints, the impact impulse $\mathbf{I}(\mathbf{x}^-)$ in (2.3) is given by a discontinuous function, which indicates the instability of motions with multiple impacts [5]. It is therefore worthwhile to investigate the important special case of orthogonal constraints [6], for which $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2$, where $\mathbf{I}_{1,2}$ are the impulses when impact occurs against each of the constraints separately. The orthogonality conditions in systems with several impact pairs are satisfied, in particular, when there are no rigid constraints among the pairs. In the case of rigid constraints, the orthogonality conditions cannot hold outside a set of zero measure in the space of constructive parameters.

Combining formulae (1.5) and (2.4), we arrive at the following proposition.

Proposition 2. If $\mathbf{x}^*(t)$ is a solution of system (1.1) which includes an impact at time $t = t'$ against two surfaces (2.1), then estimate (2.2) remains valid at time $t > t'$, where the fundamental solution matrix experiences a discontinuity in accordance with formulae (2.4), where

$$\begin{aligned} &\mathbf{B}_j(\mathbf{u},\ \mathbf{v}) = \mathbf{E}_n + \mathbf{I}_{jx} + (\mathbf{u},\ \operatorname{grad} f_j)^{-1}(\mathbf{v} - \mathbf{u} - \mathbf{I}_{jx}\mathbf{u})(\operatorname{grad} f_j)^T, \quad j = 1,2 \\ &\mathbf{B}_0 = \mathbf{E}_n + \mathbf{I}_x + (\mathbf{F}^{--},\ \operatorname{grad} f_1)^{-1}(\mathbf{F}^{++} - \mathbf{F}^{--} - \mathbf{I}_x\mathbf{F}^{--})(\operatorname{grad} f_1)^T \end{aligned} \tag{2.5}$$

3. CONDITIONS OF STABILITY IN THE FIRST APPROXIMATION

Let $\varphi : R^n \mapsto R^n$ be a mapping with a fixed point at the origin. Suppose smooth surfaces $\psi_1(\mathbf{x}) = 0$, $\psi_2(\mathbf{x}) = 0, \ldots, \psi_s(\mathbf{x}) = 0$, which have pairwise-different normals at the origin, pass through this point and divide the neighbourhood of the origin into domains $G_1, G_2, \ldots, G_r$. Let us assume that φ is continuously differentiable in the closure of each G_k and continuous (but not differentiable) at the origin, and that the following relations hold

$$\varphi(\mathbf{x}) = \begin{cases} \varphi_1(\mathbf{x}), & \text{if } \mathbf{x} \in G_1 \\ \ldots & \\ \varphi_r(\mathbf{x}), & \text{if } \mathbf{x} \in G_r \\ \varphi_{r+1}(\mathbf{x}), & \text{if } \psi_1(\mathbf{x}) = 0 \\ \ldots & \\ \varphi_{r+s}(\mathbf{x}), & \text{if } \psi_s(\mathbf{x}) = 0 \end{cases} \tag{3.1}$$

$$\varphi_k(\mathbf{x}) = \mathbf{A}_k\mathbf{x} + O(x^2), \quad k = 1, \ldots, r+s$$

At points of the boundary surfaces $\psi_j(\mathbf{x}) = 0$ the mapping φ may be discontinuous.

The system of the first approximation is obtained by replacing the surfaces of discontinuity $\psi_j(\mathbf{x}) = 0$ with their tangent planes $(\operatorname{grad} \psi_j(0), \mathbf{x}) = 0$ and dropping terms $O(x^2)$ in formulae (3.1)

$$\overline{\varphi}(\mathbf{x}) = \begin{cases} \overline{\varphi}_1(\mathbf{x}), & \text{if } \mathbf{x} \in \overline{G}_1 \\ \ldots & \\ \overline{\varphi}_r(\mathbf{x}), & \text{if } \mathbf{x} \in \overline{G}_r \\ \overline{\varphi}_{r+1}(\mathbf{x}), & \text{if } (\operatorname{grad} \psi_1(0),\ \mathbf{x}) = 0 \\ \ldots & \\ \varphi_{r+s}(\mathbf{x}), & \text{if } (\operatorname{grad} \psi_s(0),\ \mathbf{x}) = 0 \end{cases} \tag{3.2}$$

$$\overline{\varphi}_k(\mathbf{x}) = \mathbf{A}_k\mathbf{x}, \quad k = 1, \ldots, r+s$$

where $\overline{G}_k$ denotes one of the domains into which the neighbourhood of the origin is divided by the surfaces $(\mathbf{x}, \operatorname{grad} \psi_k(0))$ and $\mathbf{A}_k$ are square matrices of order n. Thus, the mapping φ_k^- is linear and $\varphi^-(\mathbf{x})$ is homogeneous (but not additive).

Let us determine in what cases the stability of system (3.1) may be inferred from an analysis of the materials $\mathbf{A}_k$. It would be a mistake to associate this conclusion directly with the positions of the eigenvalues of the matrices relative to the unit circle.

Examples. 1. Consider the piecewise-linear mapping of the two-dimensional plane into itself defined by

$$\mathbf{A}(\mathbf{x}) = \begin{cases} \mathbf{A}_+\mathbf{x} & x_2 > 0 \\ \mathbf{A}_-\mathbf{x} & x_2 \leqslant 0 \end{cases} \tag{3.3}$$

$$A_+ = \begin{Vmatrix} \alpha & 1 \\ 0 & \alpha \end{Vmatrix}, \quad A_- = \begin{Vmatrix} \alpha & 0 \\ 1 & \alpha \end{Vmatrix}, \quad \alpha \in (-1, 0)$$

The eigenvalues of each of the matrices A_+ and A_- are less than one in absolute value, indicating that each of the matrices separately is asymptotically stable. At the same time, the product

$$A_+A_- = \begin{Vmatrix} \alpha^2+1 & \alpha \\ \alpha & \alpha^2 \end{Vmatrix}$$

has an eigenvector $I^* = (1, \alpha - \alpha^3 + \ldots)^T$ with eigenvalue $\lambda^* = 1 + 2\alpha^2 - \alpha^4 \ldots > 1$. This vector I^* lies in the half-plane $x_2 < 0$ and the vector AI^* lies in the domain $x_2 > 0$.

Thus the mapping A is unstable.

2. It is a little more complicated to construct an example of an unstable continuous piecewise-linear mapping for which each of the components is asymptotically stable. Consider (3.3) with

$$A_+ = \begin{Vmatrix} 0 & -0.95 \\ 1 & -1.9 \end{Vmatrix}, \quad A_- = \begin{Vmatrix} 0 & -0.6 \\ 1 & 1 \end{Vmatrix}$$

Each of these matrices A_+ and A_- satisfies the conditions for asymptotic stability (the multipliers lie inside the unit circle). In addition, the mappings defined by them coincide on the "splicing" line $x_2 = 0$. At the same time, the mapping A^5 has an eigenvector $I^* = (1, 0)$

$$A^5(I^*) = A_-^3A_+^2(I^*) = 1.026I^*$$

Since the eigenvalue is greater than unity, the mapping is unstable.

3. An example of contrary interaction of matrices may be obtained by considering (3.3) with

$$A_+ = \mathrm{diag}\{-2;\ -0,\ 1\}, \quad A_- = \mathrm{diag}\{-0,\ 1;\ -2\}$$

In this case each of the matrices A_+ and A_- is obviously unstable, but the "composite" discontinuous mapping A is asymptotically stable. Indeed, this mapping reverses the signs of both coordinates of any vector. Consequently, the elements of the sequence $I, AI, A^2I, A^3I, \ldots$ lie alternately in the half-planes $x_2 > 0$ and $x_2 < 0$ (an exception occurs if the initial vector lies in the plane $x_2 = 0$, in which case the powers of A leave its second component equal to zero, while decreasing the first component in a geometric progression with common ratio 0.1). Stability follows from the equality

$$A_+A_- = \mathrm{diag}\{0,2;\ 0,2\}$$

We now formulate a few conditions which, if satisfied in system (3.2), have implications for the stability of the "complete" system (3.1).

Proposition 3. Suppose the norms of all the matrices A_k in formulae (3.2) do not exceed some number $q \in (0, 1)$ (in terms of some norm $\| x \|$ in R^n). Then a fixed point of the mapping (3.1) is asymptotically stable.

Indeed, in that case one can use Lyapunov's asymptotic stability theorem for mappings [7], defining the Lyapunov function to be $\| x \|$.

Proposition 4. Suppose one of the mappings $\bar{\varphi}_{k^*}$ in formula (3.2) has an eigenvector I^* in the interior of its domain of definition G_k, with an eigenvalue $\rho > 1$. Then a fixed point of the mapping (3.1) is unstable.

Proof. Consider a component φ_{k^*} of the mapping (3.1). As we know [1, 8], if a system of differential equations has ν positive characteristic values, then a ν-parameter family of solutions exists asymptotic to a singular point as $t \to -\infty$. An analogous statement holds for systems with discrete time. By a suitable choice of the parameters, one can construct a solution of the linear system with matrix A_{k^*} which lies on an invariant straight line with direction vector I^*. The solution of the complete system corresponding to these system corresponding to these parameters will asymptotically approach the straight line more rapidly than it approaches the origin. Consequently, in some neighbourhood of the origin it will lie in G_{k^*}, i.e. it will be a solution of system (3.1). Obviously, the existence of an asymptotic trajectory implies instability, since one can reach an arbitrary point of that curve from as small a neighbourhood of the origin as desired.

Corollary. The conclusion as to instability remains valid if the conditions of Proposition 4 hold for some natural power of the mapping $\bar{\varphi}$.

Indeed, the square, cube, etc. of the mapping (3.2) constitute mappings of the same type (with different subdivisions into domains of differentiability $\bar{G}_j$).

Note that if one of the mappings $\bar{\varphi}_k$ has negative eigenvalues less than –1, or imaginary eigenvalues outside the unit circle, this does not necessarily imply instability (see Example 3 in this section).

Proposition 5. Assume that, relative to some basis in R^n, the mapping (3.2) has a partitioned-triangular matrix

$$\bar{\varphi}(\mathbf{x}) = \begin{Vmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \dots & \mathbf{C}_{1q} \\ 0 & \mathbf{C}_{22} & \dots & \mathbf{C}_{2q} \\ \dots & & & \\ 0 & 0 & \dots & \mathbf{C}_{qq} \end{Vmatrix} \mathbf{x} \tag{3.4}$$

where the diagonal blocks are square matrices whose elements are the same for all components $\bar{\varphi}_k$ ($k = 1, \dots, r + s$) (the off-diagonal partitions may be different).

Then the stability of a fixed point of the mapping (3.1) may be determined from the first approximation (3.4) just as in the regular case (see Section 1).

Proof. The coefficients of the characteristic equation of the matrix (3.4) do not depend on the form of its off-diagonal partitions. Let us first assume that all these partitions vanish and that $\varphi^*(\mathbf{x})$ is a linear mapping with the partitioned-diagonal matrix obtained from (3.4) when $\mathbf{C}_{ij} \equiv \mathbf{0}$ for all $i < j$.

The mapping $\varphi^*(\mathbf{x})$ has a Lyapunov function $V(\mathbf{x})$ which is a quadratic form and satisfies the following equation [1, 7]

$$V(\varphi^*(\mathbf{x})) - V(\mathbf{x}) = \lambda V(\mathbf{x}) - \|\mathbf{x}\|^2 \tag{3.5}$$

If all the eigenvalues of $\varphi^*(\mathbf{x})$ are less than unity in absolute value, then the form V is positive definite and $\lambda = 0$, implying asymptotic stability of the simplified system. If there are eigenvalues outside the unit circle, then $\lambda > 0$ and the function V may take negative values (instability).

We will use the quadratic form V to construct a Lyapunov function for the first approximation system (3.4). To that end we make an auxiliary change of variables $\mathbf{x} \mapsto \mathbf{x}_\varepsilon$ such that the mapping (3.4), expressed in terms of the new variables, will have the same diagonal partitions as the "reduced" off-diagonal partitions. More precisely, all the elements of the upper diagonal matrices must be made less than ε in absolute value for all versions of the definition of (3.4).

To make this change of variables, it will suffice to choose scaling factors μ_k and to put $x_k + \mu_k x_{\varepsilon k}$. Indeed, when all the variables corresponding to the second diagonal partition of the matrix (3.4) are multiplied by the same constant D, all the partitions in the second row are multiplied by D and simultaneously also divided by D. Consequently, the operation leaves the matrix $\mathbf{C}_{22}$ unchanged but divided $\mathbf{C}_{12}$ by D. We can then scale the variables corresponding to the third diagonal partitions to reduce the partitions $\mathbf{C}_{13}$ and $\mathbf{C}_{23}$, etc.

We put

$$V_\varepsilon(\mathbf{x}) = V(\mathbf{x}_\varepsilon) \tag{3.6}$$

Then, by (3.5) and (3.6)

$$V_\varepsilon(\bar{\varphi}(\mathbf{x})) - V_\varepsilon(\mathbf{x}) = \lambda V_\varepsilon(\mathbf{x}) - \|\mathbf{x}_\varepsilon\|^2(1 + O(\varepsilon)) \tag{3.7}$$

For sufficiently small ε and in a sufficiently small neighbourhood of the origin, the quadratic form V_ε is a Lyapunov function for system (3.1), proving our assertion.

4. INVESTIGATION OF A SYSTEM WITH TWO IMPACT PAIRS

Let us consider a system with two interacting impact pairs of the same type

$$\ddot{x}_1 = \Phi(x_1, \dot{x}_1, t) + G(x_1 - x_2, \dot{x}_1 - \dot{x}_2), \quad \ddot{x}_2 = \Phi(x_2, \dot{x}_2, t) - G(x_1 - x_2, \dot{x}_1 - \dot{x}_2) \tag{4.1}$$

$$x_{1,2} \geqslant 0, \quad \Phi(x, \dot{x}, t + \tau) \equiv \Phi(x, \dot{x}, t), \quad G(0, 0) = 0$$

The impact in the jth pair ($j = 1, 2$) occurs when $x_j = 0, \dot{x}_j < 0$ and is described by the equality

$$\dot{x}_j^+ = -e\dot{x}_j^- \tag{4.2}$$

where $e \in (0, 1]$ is the Newton coefficient of restitution.

In particular, equations like (4.1) govern the motion of two identical spheres connected by a spring on a vibrating two-stepped base [9].

If there were no interaction between the two subsystems, the terms $\pm G$ in Eqs (4.1) would disappear. In that case the equations would also admit of solutions for which $x_1 \equiv x_2$. If one subsystem has l_τ-periodic solutions ($l \in N$), these will be preserved in the complete system if $G(0, 0) = 0$.

Let us investigate the stability of motions with one impact per period

$$x_1(t) = x_2(t) = x^*(t), \quad x^*(t+\tau) \equiv x^*(t), \quad x^*(t') = 0, \quad \dot{x}^*(t'-0) = -V \tag{4.3}$$

where the quantity $x^*(t)$ remains strictly positive for $t \neq t'$ (mod τ).

We transform system (4.1) to new variables defined by

$$z_{1,2} = (x_1 \pm x_2)/2 \tag{4.4}$$

The physical meaning of these variables is as follows: z_1 is the average value of the gap in the impact pairs and z_2 is a measure of the asymmetry of the two subsystems at the time in question.

Setting $y_{1,2}(t) = z_{1,2}(t) - x^*(t)$, we set up the equations in variations for the system in the variables (4.4)

$$\ddot{y}_1 = \Phi_x(t)y_1 + \Phi_{\dot{x}}(t)\dot{y}_1, \quad \ddot{y}_2 = (\Phi_x(t) + 2G_x)y_2 + (\Phi_{\dot{x}}(t) + 2G_{\dot{x}})\dot{y}_2 \tag{4.5}$$

$$\Phi_x(t) = \frac{\partial}{\partial x}\Phi(x^*(t), \ \dot{x}^*(t), t), \ \Phi_{\dot{x}}(t) = \frac{\partial}{\partial \dot{x}}\Phi(x^*(t), \ \dot{x}^*(t), t) \ \ G_x = \frac{\partial}{\partial x}G(0, 0),$$

$$G_{\dot{x}} = \frac{\partial}{\partial \dot{x}}G(0, 0)$$

As the variables in system (4.5) are separable, it can be solved by integrating two linear second-order systems with periodic coefficients. The resulting fundamental solution matrix $\mathbf{Y}(t_0, t)$ has the following partitioned-diagonal form in terms of the variables $y_1, \dot{y}_1, y_2, \dot{y}_2$, with diagonal partitions of second order

$$\mathbf{Y}(t_0, t) = \begin{Vmatrix} \mathbf{Y}_1(t_0, t) & 0 \\ 0 & \mathbf{Y}_2(t_0, t) \end{Vmatrix} \tag{4.6}$$

Let us calculate the impact matrices in formulae (2.4) and (2.5). Note that in the original variables

$$\mathbf{I}_1 = -\frac{1+e}{2}(0, \dot{x}_1^-, 0, 0), \ \mathbf{I}_2 = -\frac{1+e}{2}(0, 0, 0, \dot{x}_2^-)$$

Upon impact against one of the constraints, the initial conditions for impact against the other constraint remain unchanged; hence the impulses for the multiple impact are independent and the orthogonality conditions are satisfied. In terms of the variables $z_1, \dot{z}_1, z_2, \dot{z}_2$, we have

$$\begin{gathered}
\mathbf{F}^{--} = (-V, \Phi_-, 0, 0)^T, \ \ \mathbf{F}^{++} = (eV, \Phi_+, 0, 0)^T \\
\mathbf{F}^{\pm\mp} = \left(-\frac{1-e}{2}V, \ \frac{1}{2}(\Phi_+ + \Phi_-), \ \pm\frac{1+e}{2}V, \ \pm\frac{1}{2}(\Phi_+ - \Phi_-) + G_\pm\right)^T \\
\Phi_- = \Phi(0, -V, t'), \ \ \Phi_+ = \Phi(0, eV, t'), \ \ G_\pm = G(0, \pm(1+e)V) \\
I_{1,2} = -\frac{1+e}{2}(0, \dot{z}_1^- \pm \dot{z}_2^-, 0, \dot{z}_2^- \pm \dot{z}_1^-)^T, \ \operatorname{grad} f_{1,2} = (1, 0, \pm 1, 0)^T
\end{gathered} \tag{4.7}$$

Calculating $\mathbf{B}$ from formulae (2.4), we obtain expressions in which the first, second and fourth columns of all three versions are the same

$$\mathrm{col}_1 = (-e, -\xi, 0, 0), \quad \mathrm{col}_2 = (0, -e, 0, 0), \quad \mathrm{col}_4 = (0, 0, 0, -e) \tag{4.8}$$

The third column of **B** depends on the order of the impacts and is as follows:

$$\mathrm{col}_3 = (0, -\eta_+ G_+, -\xi - \eta_- G_+, 0), \quad \text{for } \Delta_2 t < \Delta_1 \tag{4.9}$$

$$\mathrm{col}_3 = (0, -\eta_+ G_-, -\xi + \eta_- G_-, 0), \quad \text{for } \Delta_2 t > \Delta_1 \tag{4.10}$$

$$\mathrm{col}_3 = (-(1+e), -\xi, 1, 0), \quad \text{for } \Delta_2 t = \Delta_1 \tag{4.11}$$

where $\xi = (\Phi_+ + e\Phi_-)/V$, $\eta^{\pm} = (1 \pm e)/V$.

Note that the order of the impacts against the constraint in this problem is determined by the sign of y_2 at time $t = t'$. Upon simultaneous impact $y_2 = 0$, and the elements in the third column of the matrix $\mathbf{B}_0$ do not affect the value of $\mathbf{Y}(t_0, t' + 0)$. If this column is set equal to half the sum of the columns (4.9) and (4.10), the result is unchanged. At the same time, by (4.6), we conclude that the monodromy matrix satisfies the assumptions of Proposition 5 if $G_- = -G_+$. This condition implies that the force of interaction between the elements of the two impact pairs of symmetric. If $G^{\pm} = 0$, the Poincaré mapping φ is differentiable; otherwise it is piecewise-differentiable.

By Proposition 5, the stability conditions for the symmetric periodic motions under discussion may be divided into two groups. The first group corresponds to motion of one impact pair without any connection with the other; in that situation the monodromy matrix is the product of the matrix $\mathbf{Y}_1(t_0, t_0 + \tau)$ and the second-order minor at the upper left corner of the impact matrix. The second group describes the effect on stability of a connection between the pairs, with the monodromy matrix equal to the product of $\mathbf{Y}_2(t_0, t_0 + \tau)$ and the second-order minor at the lower right corner of the impact matrix.

One of the basic models of the theory of vibro-impact systems is a particle on a vibrating base [10]. In such a system

$$\Phi(x, \dot{x}, t) = -g - \ddot{h}(t)$$

where g is the acceleration due to gravity and $h(t)$ is the ordinate of the supporting surface in some inertial system of coordinates. The conditions for the existence of a motion with one impact per period $l\tau$ are

$$V = \frac{gl\tau}{1+e}, \quad \dot{h}(t') = \frac{1-e}{2(1+e)} gl\tau \tag{4.12}$$

Since $\Phi_x = \Phi_{\dot{x}} \equiv 0$, it is not difficult to write down a solution of system (4.5): in formula (4.6)

$$\mathbf{Y}_1(t', t' + l\tau) = \begin{Vmatrix} 1 & l\tau \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{Y}_2(t', t' + l\tau) = \begin{Vmatrix} c_l + ks_l & s_l \\ -c^2 s_l & c_l - ks_l \end{Vmatrix} \tag{4.13}$$

$$c_l = \exp(-kl\tau)\cos(\delta l\tau), \quad s_l = \delta^{-1}\exp(-kl\tau)\sin(\delta l\tau), \quad \delta = \sqrt{c^2 - k^2}$$

$$k = -G_{\dot{x}}(0,0), \quad c^2 = -G_x(0,0)/2$$

The first group of stability conditions consists of the Schur inequalities for the second-order characteristic polynomial

$$|\operatorname{Tr} \mathbf{C}_{11}| < 1 + \det \mathbf{C}_{11} < 2, \quad \mathbf{C}_{11} = \mathbf{B}_{11}\mathbf{Y}_1(t', t' + l\tau) \tag{4.14}$$

where $\mathbf{B}_{11}$ is the second-order minor in the upper left corner of the matrices (4.8).

If conditions (4.12) are taken into consideration, inequalities (4.14) become

$$-\frac{2(1+e^2)}{(1+e)^2} g < \ddot{h}(t') < 0 \tag{4.15}$$

The second group of stability conditions is analogous in form to (4.14), but with the matrix $\mathbf{C}_{11}$ replaced by $\mathbf{C}_{22} = \mathbf{B}_{22}\mathbf{Y}_2(t', t' + l\tau)$. Performing the necessary computations, we finally obtain the following inequality

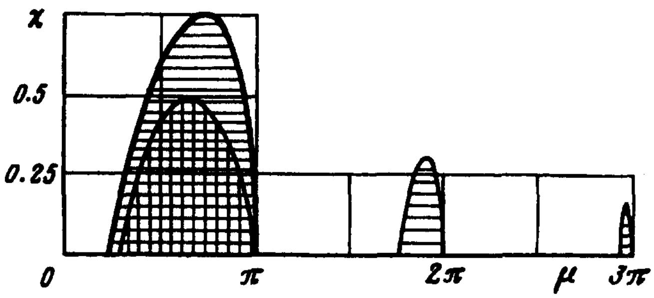

Fig. 2.

$$\left|\left(\frac{(1+e)^2}{k\tau}\left(1+\frac{\ddot{h}(t')}{g}\right)-\frac{1-e}{V}G_+\right)s_l-2ec_l\right|<1+e^2\exp(-2kl\tau) \tag{4.16}$$

The satisfaction of both conditions (4.15) and (4.16) guarantees the motion to be asymptotically stable; but if the sign of at least one of these inequalities is reversed, the motion is unstable.

In particular, suppose the motion of the base is governed by a harmonic law $h(t) = \varepsilon \sin \omega t$ ($\omega = 2\pi/\tau$), the impacts are absolutely elastic, and the spring is linear (in that case the determinant of the matrix C_{11} is equal to 1 and inequalities (4.15) are necessary but not sufficient for stability). It then follows from the periodicity condition (4.12) that there are two types of periodic solutions: those for which $\sin \omega t' = 1$ or $\sin \omega t' = -1$. The second of these solutions does not satisfy inequality (4.15), and hence it is unstable. The stability conditions for the first motion take the form

$$\Gamma < 1, \quad \left|2\frac{\sin(\delta l\tau)}{\delta l\tau}(1-\Gamma)-\cos(\delta l\tau)\right| < \operatorname{ch}(kl\tau) \tag{4.17}$$

where $\Gamma = \varepsilon\omega^2/g$ is the intensity of the excitation.

The domain (4.17) is shown in Fig. 2 in the plane of the parameters $\mu = \delta/\tau$, $\chi = k/\tau$ for $\Gamma = 0.05$. The instability zones, shown hatched in the figure, have the form of "teeth" whose bases lie on the abscissa axis and are adjacent on their right to the points $m\pi$, $m \in N$. The width and height of the teeth decrease as m increases. We have appealed here to the conclusion, obtained in [9], that for an ideal spring of small stiffness the system is stable, but as the stiffness increases it becomes unstable.

As the parameter Γ increases, the instability zones become smaller in size.

If $e < 1$, then, as computations have shown, the instability zones become smaller. Thus, when $e = 0.7$, $\ddot{h}(t')/g = -0.05$, only one of the "teeth" remains (cross-hatched in Fig. 2).

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